FEATURES OF AXISYMMETRICAL PLASMA FLOWS IN A NARROW FLUX TUBE

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UDC 533.9

1. Introduction. A model for flow in a narrow flux tube was suggested in [1] in order to study the properties of magnetohydrodynamic plasma flows in channels. Steady-state axially-symmetric flow of a nonviscous non-heat conducting ideally conducting medium in a transverse natural magnetic field is described by the equations

$$\rho v s = \text{const}; \tag{1.1}$$

$$\frac{v^2}{2} + \frac{\gamma p/\rho}{\gamma - 1} + \frac{H^2}{4\pi\rho} = \text{const}; \tag{1.2}$$

$$H/\rho r = \text{const}; \tag{1.3}$$

$$p/\rho^\gamma = \text{const}. \tag{1.4}$$

Here $s = 2\pi fr$ is tube cross-sectional area; r is its average radius; f is transverse dimension (Fig. 1). Equations (1.2)-(1.4) describe the change in flux parameters along the trajectory $r = r(z)$. Generally speaking Eq. (1.1) contains values of ρ and v averaged over the tube cross section, although in view of the assumed smallness of f it is possible to consider them coincident with values of ρ and v in trajectory r(z).

In studying flow in channels there is extensive use of a hydraulic (quasi-uniform) approximation in which all MHD-values are averaged over the cross section. A review of the results may be found for example in [2]. An undoubted virtue of the hydraulic approximation is the possibility of considering the effect on flow of various physical factors (dissipation processes, an external electromagentic field, etc.). In this model channel geometry affects flow similar to normal gas dynamics. Another situation arises in studying flow in a narrow flux tube. One one hand this model rests on the results of equations for an ideal magnetic hydrodynamic (Eqs. (1.2)-(1.4) are precise conservation rules which are fulfilled along the trajectory). Therefore, if it is very difficult for example to consider dissipation factors. Equations are given in [3] for a plasma with finite conductivity, but with the stringent assumptions that the trajectory coincides with equipotentials (with finite conductivity that is automatically fulfilled). On the other hand, in Eqs. (1.1)-(1.4) there are precise MHD-values in which the trajectory is of arbitrary shape. This makes it possible within the scope of a model for flow in a narrow tube to consider essentially two-dimensional effects. It is shown in [4] that with certain assumptions about the shape of narrow tubes it is possible to obtain conditions for absence in a channel of electric current eddies expressed in the form of limitations on local flow parameters.

From a mathematical point of view the 'quasi-two-dimensionality' of the flow model in a narrow tube is due to the fact that all the values in Eqs. (1.1)-(1.4) are considered at points of two-dimensional space with coordinates (z, r(z)). From a physical viewpoint this model considers an important situation which in principle it is not possible to consider with averaging over the cross section: along a trajectory of arbitrary shape there is a change not only in the intensity of the magnetic field, but also in the length of its force lines. In other words, in equations of motion there is consideration not only of the magnetic pressure gradient, but also of the term $\sim(\mathbf{H}\nabla)\mathbf{H}$, which means that the field performs work in shortening its force lines [1].

As a well-known example we consider the question of a change-over in terms of signal velocity. By differentiating Eqs. (1.1)-(1.4) and excluding $d\rho$, dp, dH, we obtain [1]

$$\left(v^2 - c_m^2\right)\frac{dv}{v} = c_s^2\frac{d\,(fr)}{fr} + c_a^2\frac{d\,(f/r)}{f/r} \tag{1.5}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 3-10, September-October, 1991. Original article submitted June 19, 1990.

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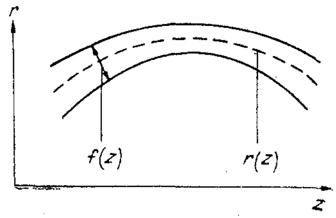

Fig. 1

($c_s^2 = \gamma p/\rho$, $c_a^2 = H^2/(4\pi\rho)$, $c_m^2 = c_s^2 + c_a^2$). In bent tubes ($r \neq$ const) a change-over in terms of local signal velocity occurs with $c_s^2 \gg c_a^2$ (gas dynamic flow) at the point of a minimum for function fr (cross-sectional area), and with $c_s^2 \ll c_a^2$ (flow in a strong magnetic field) at a point of the minimum of function f/r. In a tube with r = const or in the case of plane flow (which formally corresponds to $r \to \infty$) the nature of flow is determined by the behavior of f, i.e., in fact the cross-sectional area, and presence of a magnetic field does not affect the position of the point of a change-over in terms of signal velocity.

Features are considered in the present work for flow of a plasma in a narrow flux tube which arises as a result of curvature of its central line. In addition, the question is studied of existence of steady-state flow in narrow tubes.

2. Features of Flow in Bent Tubes. We rewrite (1.5) in the form

$$(M^2 - 1)\frac{dv}{v} = \frac{df}{f} - \frac{1-\alpha}{1+\alpha}\frac{dr}{r}. \tag{2.1}$$

Here $M = v/c_m$ is magnetic Mach number; $\alpha = \gamma\beta/2$; $\beta = 8\pi p/H^2$. According to Eqs. (1.1)-(1.4) each MHD-value may be expressed in terms of functions $r(z)$, $f(z)$, and constants of equations which are combinations of input values ρ_0, v_0, H_0, p_0, r_0, f_0. The same is also correct for function $\beta(z)$. Consequently, the position of the critical cross section (in which a change-over is possible in terms of signal velocity) in the general case is determined not only by tube geometry (functions $r(z)$ and $f(z)$), but also by flux input parameters. this assertion may be illustrated by the following example. Since from (1.3) and (1.4) it follows that

$$\beta\rho^{2-\gamma}r^2 = \text{const}, \tag{2.2}$$

then with $\gamma = 2\beta = \beta_0(r_0/r)^2$ it is possible to write the right-hand part of (2.1) in the form of a total differential. Finally we find that with $\gamma = 2$ a change-over in terms of signal velocity occurs in section z_*, where

$$rf/(r^2 + \beta_0 r_0^2) = \min. \tag{2.3}$$

It is evident that (2.3) determines the implicit dependence $z_*(\beta_0)$. Here and subsequently indices 0 and * denote values which relate to the initial and critical cross sections, respectively. However, if functions $r(z)$ and $f(z)$ are such that fr and f/r behave in the same way and have a minimum at the same point z^0, then according to (1.5) z_* agrees with z^0 and it does not depend on β_0.

In gas-dynamic flow the directions for the change in density and velocity in a narrow tube are different. In MHD-flow they may coincide. In fact, by differentiating Eqs. (1.1)-(1.4) and excluding dp, dv, dH, we have the relationship

$$(M^2 - 1)\frac{d\rho}{\rho} = \left(\frac{2}{1+\alpha} - M^2\right)\frac{dr}{r} - M^2\frac{df}{f}, \tag{2.4}$$

which we shall consider together with (2.1). It can be seen that in normal gas dynamics ($\alpha \to \infty$) always $d\rho dv < 0$. This is also fulfilled for MHD-flow in a tube with r = const. In a tube with $r \neq$ const a situation is possible when $d\rho dv > 0$. Accelerating regimes with densification are observed in calculations for steady-state two-dimensional plasma flows in channels [5]. In [1] such regimes are called anomalous. We clarify conditions for realizing them.

Multiplying (2.1) and (2.4) we find that

$$(M^2-1)^2\frac{d\rho\, dv}{\rho v}=\left[\left(\frac{2}{1+\alpha}-M^2\right)\frac{dr}{r}-M^2\frac{df}{f}\right]\left[\frac{df}{f}-\frac{1-\alpha}{1+\alpha}\frac{dr}{r}\right].$$

Consequently in an anomalous regime there is simultaneous fulfillment of the inequalities

$$\left(\frac{2}{1+\alpha}-M^2\right)\frac{dr}{r}-M^2\frac{df}{f}>0,\quad \frac{df}{f}-\frac{1-\alpha}{1+\alpha}\frac{dr}{r}>0, \tag{2.5}$$

or

$$\left(\frac{2}{1+\alpha}-M^2\right)\frac{dr}{r}-M^2\frac{df}{f}<0,\quad \frac{df}{f}-\frac{1-\alpha}{1+\alpha}\frac{dr}{r}<0. \tag{2.6}$$

The condition for compatibility of (2.5) is:

$$\frac{1-\alpha}{1+\alpha}\frac{dr}{r}<\left(\frac{2/M^2}{1+\alpha}-1\right)\frac{dr}{r}$$

or

$$(M^2-1)dr<0. \tag{2.7}$$

Similarly the compatibility condition for (2.6) is:

$$(M^2-1)dr>0. \tag{2.8}$$

Taking account of (2.5)-(2.8) it is possible to write conditions for the anomalous behavior of density:

a) in an acceleration regime ($dv > 0$) $dr < 0$,

$$(M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r}<(M^2-1)\frac{df}{f}<(M^2-1)\left(\frac{2/M^2}{1+\alpha}-1\right)\frac{dr}{r};$$

b) in a retardation regime ($dv < 0$), $dr > 0$.

$$(M^2-1)\left(\frac{2/M^2}{1+\alpha}-1\right)\frac{dr}{r}<(M^2-1)\frac{df}{f}<(M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r}.$$

Thus, independent of the nature of flow, i.e., sub-signal ($M < 1$) or super-signal ($M > 1$), anomalous acceleration is only possible in ascending trajectories ($dr > 0$).

In normal hydrodynamics acceleration or retardation is clearly defined by the direction of the change in thermal Mach number Maxa $M_s = v/c_s$. In MHD-flow the direction of increase in M, M_s, and parameter M_a, inverse to Alfven number ($M_a = v/c_a$), along the trajectory may not coincide with the direction of an increase in velocity. In fact, from (1.1)-(1.4) there follow the relationships

$$(M^2-1)\frac{dM}{M}=\left[\frac{M^2}{2}\frac{1+(\gamma-1)\alpha}{1+\alpha}+1\right]\frac{df}{f}- \tag{2.9}$$
$$-\left[\frac{M^2}{2}\frac{1-(\gamma-1)\alpha}{1+\alpha}+\frac{1-\alpha}{1+\alpha}+\frac{(\gamma-2)\alpha}{(1+\alpha)^2}\right]\frac{dr}{r};$$

$$(M^2-1)\frac{dM_s}{M_s}=\left(1+\frac{\gamma-1}{2}M^2\right)\frac{df}{f}-\left(\frac{\gamma-\alpha}{1+\alpha}-\frac{\gamma-1}{2}M^2\right)\frac{dr}{r}; \tag{2.10}$$

$$(M^2-1)\frac{dM_a}{M_a}=\left(\frac{M^2}{2}+1\right)\frac{df}{f}-\left(\frac{M^2}{2}+\frac{1-2\alpha}{1+\alpha}\right)\frac{dr}{r}. \tag{2.11}$$

Without giving the calculations similar to those previously we write conditions for anomalous (in the sense indicated) behavior of M, M_s, M_a:

a) in an acceleration regime ($dv > 0$) $dM < 0$ with $(\gamma - 2)dr < 0$ and

$$(M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r}<(M^2-1)\frac{df}{f}<$$
$$<(M^2-1)\frac{M^2[1-(\gamma-1)\alpha]+2\dfrac{1+(\gamma-2)\alpha-\alpha^2}{1+\alpha}}{M^2[1+(\gamma-1)\alpha]+2(1+\alpha)}\frac{dr}{r},$$

$dM_s < 0$ with $dr < 0$ and

$$(M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r} < (M^2-1)\frac{df}{f} < (M^2-1)\frac{\frac{\gamma-\alpha}{1+\alpha}-\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dr}{r},$$

$dM_a < 0$ with $dr > 0$ and

$$(M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r} < (M^2-1)\frac{df}{f} < (M^2-1)\frac{M^2+2\frac{1-2\alpha}{1+\alpha}}{M^2+2}\frac{dr}{r};$$

b) in a retardation regime ($dv < 0$) $dM > 0$ with $(\gamma - 2)dr > 0$ and

$$(M^2-1)\frac{M^2[1-(\gamma-1)\alpha]+2\frac{1+(\gamma-2)\alpha-\alpha^2}{1+\alpha}}{M^2[1+(\gamma-1)\alpha]}\frac{dr}{r} <$$
$$< (M^2-1)\frac{df}{f} < (M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r},$$

$dM_s > 0$ with $dr < 0$ and

$$(M^2-1)\frac{\frac{\gamma-\alpha}{1+\alpha}-\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\frac{dr}{r} < (M^2-1)\frac{df}{f} < (M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r},$$

$dM_a > 0$ with $dr < 0$ and

$$(M^2-1)\frac{M^2+2\frac{1-2\alpha}{1+\alpha}}{M^2+2}\frac{dr}{r} < (M^2-1)\frac{df}{f} < (M^2-1)\frac{1-\alpha}{1+\alpha}\frac{dr}{r}.$$

It is noted that with $\gamma = 2$ always $dMdv \geq 0$.

Thus, with $\gamma < 2$ in ascending trajectories acceleration clearly specifies an increase in M_s, and retardation a reduction in M and M_a. In descending trajectories acceleration clearly specifies an increase in M and M_a, and retardation specifies a reduction in M_s.

Finally we consider a narrow tube with a constant transverse size f = const. From (2.1) we obtain

$$(M^2-1)\frac{dv}{v} = \frac{\alpha-1}{\alpha+1}\frac{dr}{r}. \tag{2.12}$$

The sign of acceleration at each point is determined by the nature of flow (sub-signal or super-signal), by the direction of the increase in function r(z), and by the value of β at this point. A change-over of function $\beta(z)$ through $2/\gamma$ with a uniform change in r(z) is accompanied by a change in the sign of acceleration, i.e., appearance of a local extremum in velocity. From (2.9) we obtain

$$(M^2-1)\frac{dM}{M} = -\left[\frac{M^2}{2}\frac{1-(\gamma-1)\alpha}{1+\alpha} + \frac{1+(\gamma-2)\alpha-\alpha^2}{(1+\alpha)^2}\right]\frac{dr}{r}. \tag{2.13}$$

From (2.2) taking account of (2.4) we find that

$$(M^2-1)\frac{d\beta}{\beta} = \left[2\frac{\gamma-1+\alpha}{1+\alpha} - \gamma M^2\right]\frac{dr}{r}. \tag{2.14}$$

From expressions (2.12)-(2.14) it is possible to determine the direction of increase in v, M, and β at any point in relation to their values at this point and the direction for the change in function r(z). Results of studies for $3/2 < \gamma < 2$ are presented graphically in Figs. 2 and 3 with $dr > 0$ and $dr < 0$, respectively, and lines I and II correspond to

$$M^2 = 2\frac{(\gamma-2)\alpha+1-\alpha^2}{(\gamma-2)\alpha+(\gamma-1)\alpha^2-1}, \quad M^2 = \frac{2}{\gamma}\frac{\gamma-1+\alpha}{1+\alpha}.$$

Arrows indicate the direction of change in M^2 and α (i.e., β). In Fig. 2 in a square bounded by straight lines M = 0 and 1, α = 0 and 1, we have $dM > 0$, $d\beta < 0$, etc.

For definiteness we consider a tube with $d^2r/dz^2 < 0$ at whose inlet the flow is sub-signal. With $\alpha_0 < 1$ it accelerates, and at the point of a maximum for function r(z) a

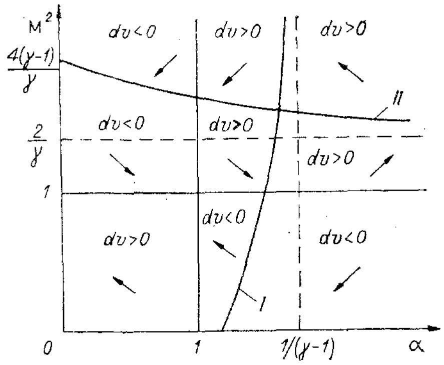

Fig. 2

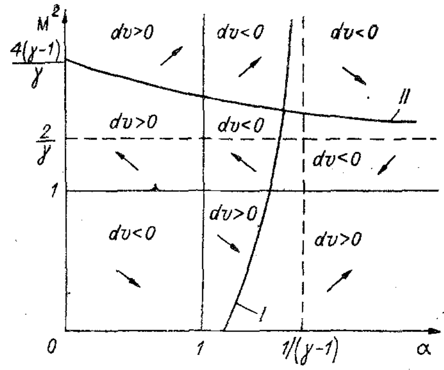

Fig. 3

change-over is possible in terms of signal velocity. If a change-over occurs, then the flow continues to accelerate, but starting from some point α increases and with $\alpha = 1$ the velocity reaches a maximum. With $\alpha_0 > 1$ the sub-signal flow is retarded. As can be seen from Figs. 2 and 3 depending on initial values of M_0 and α_0 such flow regimes are possible in which M increases in a retarding flow, and at the point of a maximum of function r(z) a changeover is possible through M = 1. Other flow regimes are also considered in a similar way on the basis of Figs. 2 and 3.

3. Existence of Steady-State Flows. In normal gas dynamics conditions for existence of steady-state flow in a quasi-uniform channel (Laval nozzle) are expressed by limitations on initial parameter M_{s0}:

$$0 < M_{s0} \leqslant M_{s0}^{-}, \quad M_{s0} \geqslant M_{s0}^{+}, \quad M_{s0}^{-} < 1 < M_{s0}^{+},$$

M_{s0}^{-} and M_{s0}^{+} are determined by the condition for a change-over in terms of sound velocity c_s in a critical (minimum) cross section. As shown above, in magnetohydrodynamics for a narrow tube of arbitrary shape the position of the critical cross section is previously unknown, and therefore the question of existence of steady-state flow should be considered separately.

We rewrite the set of equations (1.1)-(1.4) in dimensionless form taking input values as measurement units for the corresponding values ρ_0, c_{m0}, H_0, p_0, s_0, r_0:

$$\rho v s = M_0, \quad \frac{v^2}{2} + \frac{p/\rho}{\gamma-1}\frac{\alpha_0}{1+\alpha_0} + \frac{H^2}{\rho}\frac{1}{1+\alpha_0} = \frac{M_0^2}{2} + \frac{1+\alpha_0/(\gamma-1)}{1+\alpha_0}, \qquad (3.1)$$
$$H/\rho r = 1, \; p/\rho^{\gamma} = 1.$$

Here $s = fr$; $\alpha_0 = \gamma\beta_0/2$. With this selection of units flow in a tube with prescribed geometry is determined by dimensionless parameters M_0 and β_0. We find with what values of these parameters a solution of system (3.1) exists in each tube cross section (i.e., steady-state flow exists described by Eqs. (3.1)).

Excluding v, H, p from (3.1) we obtain an equation for density:

$$\rho^3 r^2 + \rho^{\gamma+1}\frac{\alpha_0}{\gamma-1} - \left[\frac{M_0^2}{2}(1+\alpha_0) + \frac{\alpha_0}{\gamma-1} + 1\right]\rho^2 + \frac{M_0^2}{2s^2}(1+\alpha_0) = 0. \qquad (3.2)$$

Since all of the rest of the MHD-values are clearly expressed in terms of ρ it is necessary to clarify conditions for existence of roots for Eq. (3.2).

We consider the left-hand part of (3.2) as a function of $F(\rho)$ with fixed r, s, M_0, α_0. It is easy to be sure that $F(\rho)$ has a minimum with $\rho = \rho_m$, and the value of ρ_m is determined from the equation

$$3\rho_m r^2 + \alpha_0\frac{\gamma+1}{\gamma-1}\rho_m^{\gamma-1} = M_0^2(1+\alpha_0) + 2\left(\frac{\alpha_0}{\gamma-1} + 1\right). \qquad (3.3)$$

With any value of r, M_0, α_0 this equation has a single positive root. For existence of roots of Eq. (3.2) it is necessary and sufficient that the following condition is fulfilled

$$F(\rho_m) \leqslant 0. \qquad (3.4)$$

If $F(\rho_m)$ is a negative value, then Eq. (3.2) has the roots ρ_- and ρ_+: $\rho_- < \rho_m < \rho_+$, and $F'(\rho_-) < 0$, $F'(\rho_+) > 0$. By using an expression for M in dimensionless values

$$M^2 = \frac{(1+\alpha_0) M_0^2}{\rho^2 s^2 (\alpha_0 \rho^{\gamma-1} + \rho r^2)},$$

we present (3.2) in the form

$$\frac{1}{2}(M^2-1)\left(\rho^3 r^2 + \frac{\alpha_0}{2}\rho^{\gamma+1}\right) + \frac{1}{2}\rho F'(\rho) = 0,$$

whence it follows that ρ_+ corresponds to sub-signal flow, and ρ_- corresponds to super-signal flow. Thus, solving Eq. (3.2) at each point z we obtain two functions: $\rho_+(z)$ and $\rho_-(z)$. At point z_*, where $F(\rho_m) = 0$ and $M = 1$, their values coincide. Evidently the behavior of density in the tube with $z \leq z_*$ is clearly described by one of these functions, the choice of which determines M . With $z > z_*$ both solutions have a physical meaning. One of them describes trans-signal flow, and the other regime in which there is no change-over in terms of signal velocity.

We clarify the meaning of condition (3.4). We express M_0^2 from (3.3) and we place it in $F(\rho_m)$. Condition (3.4) is equivalent to

$$\Phi(\rho_m) \geqslant 0, \tag{3.5}$$

$$\Phi(\rho_m) = \rho_m^3 r^2 + \alpha_0 \rho_m^{\gamma+1} - \frac{3r^2}{s^2}\rho_m - \alpha_0 \frac{\gamma+1}{\gamma-1}\frac{\rho_m^{\gamma-1}}{s^2} + 2\frac{1+\alpha_0/(\gamma-1)}{s^2}.$$

Function $\Phi(x)$ has a minimum point $x = 1/s$. Condition (3.5) is fulfilled for any ρ_m if $\Phi(1/s) \geq 0$ or

$$\frac{r^2}{s} + \frac{\alpha_0}{\gamma-1}\frac{1}{s^{\gamma-1}} \leqslant 1 + \frac{\alpha_0}{\gamma-1}.$$

If this equality is not fulfilled, i.e.

$$\frac{r^2}{s} + \frac{\alpha_0}{\gamma-1}\frac{1}{s^{\gamma-1}} > 1 + \frac{\alpha_0}{\gamma-1}, \tag{3.6}$$

then (3.5) satisfies the values of ρ from the ranges

$$0 < \rho_m \leqslant \rho_m^-, \quad \rho_m \geqslant \rho_m^+,$$

where ρ_m^- and ρ_m^+ are roots of the equation $\Phi(\rho_m) = 0$, and for them there is fulfillment of

$$\rho_m^- < \frac{1}{s} < \rho_m^+. \tag{3.7}$$

Evidently $\rho_m > \rho_m^0$ makes sense where ρ_m^0 is found from (3.3) with $M_0 = 0$:

$$\frac{3}{2}\rho_m^0 r^2 + \frac{\alpha_0}{2}\frac{\gamma+1}{\gamma-1}(\rho_m^0)^{\gamma-1} = 1 + \frac{\alpha_0}{\gamma-1}.$$

Taking account of (3.6) we obtain

$$\frac{3}{2}\frac{r^2}{s} + \frac{\alpha_0}{2}\frac{\gamma+1}{\gamma-1}\frac{1}{s^{\gamma-1}} > 1 + \frac{\alpha_0}{\gamma-1}$$

and consequently $\rho_m^0 < 1/s$. Since $\Phi(\rho_m^0) > 0$, it is possible to be certain immediately that $\rho_m^0 < \rho_m^-$. Thus, acceptable values of ρ_m are contained in the ranges $\rho_m^0 < \rho_m \leq \rho_m^-$, $\rho_m \geq \rho_m^+$. Taking account of (3.3) this means that with prescribed β_0 acceptable values of M_0 are contained in the ranges $0 < M_0 \leq M_0^-$, $M_0 \geq M_0^+$ (M_0^- and M_0^+ are obtained with substitution in (3.3) of ρ_m^- and ρ_m^+).

By carrying out similar consideration in each tube cross section* we obtain functions

*With r = const according to (3.3) ρ_m, and consequently M^- and M^+, are identical in all cross sections. In order to find limiting values of M_0 it is sufficient to solve the equation $\Phi(\rho_m) = 0$ in a known critical cross section s_{min}.

$M_0^-(z)$ and M_0^+. Apparently if M_0 is selected from the ranges

$$0 < M_0 \leqslant M_{\min}^-, \quad M_0 \geqslant M_{\max}^+$$
$$\left(M_{\min}^- = \min_z M_0^-(z) = M_0^-(z_-), \quad M_{\max}^+ = \max_z M_0^+(z) = M_0^+(z_+)\right),$$

then a solution for Eq. (3.2) exists in any cross section. If M_0 exactly equals one of the limiting values $M_{\min}^-$ or $M_{\max}^+$, then correspondingly at points z_- or z_+ values of ρ_m^- or ρ_m^+ satisfy Eq. (3.2). This means that $M(z_-) = 1$ or $M(z_+) = 1$. It is noted that the position of the point of change-over in terms of signal velocity appear to be connected with the complete collection of input values of dimensional MHD-values: ρ_0, p_0, v_0, H_0.

Thus, in a tube with prescribed geometry and fixed input parameter β_0 there is no steady-state flow regime in which M_0 takes a value of some open range depending on β_0 and geometry. It is possible to show that for a narrow tube in which functions fr and f/r have minima at the same point this range contains $M_0 = 1$. In fact, according to (1.5) in such a tube a sub-signal flow at the inlet accelerates and super-signal flow slows down. By placing $M_{\min}^-$ and $M_{\max}^+$ in the first equations of set (3.1) and using (3.7) we obtain

$$v_- > M_{\min}^-, \quad v_+ < M_{\max}^+.$$

Here v_- and v_+ are velocities in the critical cross section relating to values of density β_0. Since the input Mach number is a dimensionless value of inlet velocity, then these inequalities imply β_0. Finally, we mention special cases when equations which determine limiting values of Mach number may be written explicitly. From expressions (2.9)-(2.11) we have (see also for example [6]) with $\beta \gg 1$

$$\frac{\left(1+\frac{\gamma-1}{2}M_s^2\right)^{(\gamma+1)/(\gamma-1)}}{M_s^2} = \frac{\left(1+\frac{\gamma-1}{2}M_{s0}^2\right)^{(\gamma+1)/(\gamma-1)}}{M_{s0}^2}s^2,$$

with $\beta \ll 1$

$$\frac{(M_a^2+2)^3}{M_a^2} = \frac{(M_{a0}^2+2)^3}{M_{a0}^2}(f/r)^2,$$

with $\gamma = 2$

$$\frac{(M^2+2)^3}{M^2} = \frac{(M_0^2+2)^3}{M_0^2}\left[\frac{(1+\beta_0)s}{r^2+\beta_0}\right]^2.$$

We recall that measurement units for values f, r, and s are their input values. In these special cases the position of the critical cross section is known, and therefore equations for limiting input values M_{s0}, M_{a0}, M_0 have the form

$$\frac{\left(1+\frac{\gamma-1}{2}M_{s0}^2\right)^{(\gamma+1)/(\gamma-1)}}{M_{s0}^2} = \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(\gamma-1)}\frac{1}{s_*^2}, \quad s_* = \min_z s; \tag{3.8}$$

$$\frac{(M_{a0}^2+2)^3}{M_{a0}^2} = \frac{27}{(f_*/r_*)^2}, \quad \frac{f_*}{r_*} = \min_z \frac{f}{r}; \tag{3.9}$$

$$\frac{(M_0^2+2)^3}{M_0^2} = 27\left[\frac{\beta_0+r_*^2}{s_*(\beta_0+1)}\right]^2. \tag{3.10}$$

In the last equation r_* and s_* satisfy (2.3). We can be sure that each of these equations has two roots bounding the ranges of impermissible values of M_{s0}, M_{a0}^2, M_0, and the ranges contain unity. If the input parameters take their limiting values, then velocity in the critical cross section reaches values of the local signal velocity. Therefore, equalities (3.8)-(3.10) are necessary conditions for a trans-signal change-over in each of the particular cases considered.

In conclusion it is noted that input parameter M_0 may be expressed in terms of 'integral' (for the given narrow tube) characteristic of flow: plasma mass flow rate $\dot{m} = \rho_0 v_0 s_0$, total current occurring in the tube $I = H_0 c r_0/2$, and the difference in potential between trajectories bounding the tube $U = H_0 v_0 f_0/c$. By using these values we find that

$$M_0^2 = \frac{c^4 r_0^2}{4 f_0^2} \frac{\dot{m} U}{I^3 (1 + \gamma\beta_0/2)}. \quad (3.11)$$

Thus, with fixed β_0 the condition for trans-signal change-over determining M_0 in relation to β_0 and tube geometry is a condition for combination of integral values $Q = \dot{m}U/I^3$. In particular we consider the case of $\gamma = 2$. According to (3.10) with r = const the dependence of limiting value M_0 on β_0 disappears, and the limiting value of Q_m is a linear function of β_0. If a change in r cannot be ignored, then with $f_* \ll 1$ from (3.10) we obtain approximate equations

$$M_0^2 = \frac{8}{27}\left[\frac{s_*(1+\beta_0)}{r_*^2 + \beta_0}\right]^2, \quad M_0^2 = 3\sqrt{3}\,\frac{\beta_0 + r_*^2}{s_*(\beta_0 + 1)},$$

the first of which relates to sub-signal flow at the inlet, and the second the super-signal flow. By substituting them in (3.11) we find correspondingly

$$Q_m^- = \frac{32 f_0^2}{27 c^4 r_0^2} \frac{s_*^2 (1+\beta_0)^3}{(r_*^2 + \beta_0)^2}, \quad Q_m^+ = \frac{12\sqrt{3} f_0^2}{c^4 r_0^2} \frac{\beta_0 + r_*^2}{s_*}.$$

Here s_* and r_* are as before dimensionless values. Let them be independent of β_0. Then $Q_m^+(\beta_0)$ is a linear function; $Q_m^-(\beta_0)$ is an increasing function if $r_*^2 > 2/3$, and it has a minimum with $\beta_0 = 2/3 - r_*^2$, if $r_*^2 < 2/3$.

The author thanks K. V. Brushlinskii for reading the manuscript and making many useful comments.

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